# Start Math Right

## The Journey to Mathematics Excellence

Joan A. Cotter, Ph.D.

with Kathleen Cotter Clayton

Library of Congress Cataloging-in-Publication Data is available upon request.
ISBN 978-1-964534-00-8

Activities for Learning, Inc.
321 Hill Street, Hazelton, North Dakota 58544 USA
Printed in India

First paperback edition March 2025

# Contents

only one-third of the states use either one of them. A common criticism is that most test questions are phrased in sentences, requiring good reading skills. This is especially difficult for those with learning disabilities or those learning English as a second language. One part of a third-grade sample question reads, "What is the total number of pennies Nolan has after he adds the 10 pennies from his pocket to the jar?" This question has 21 words and scores at grade 8.4 for readability according to the Gunning Fog Index.

## Introduction Summary

A tragedy of mathematics education today in the U.S. is that about half of the adults have math anxiety. They were not born with this disorder. They learned it! Perhaps educators need a version of the statement from the Hippocratic oath: "Do no harm to the patient." An oath to "Do no harm to the learning child" could benefit many.

Today, much more is known about how children learn. We know that rote memorization is temporary and needs frequent review. Twenty percent of children simply cannot memorize, yet they can learn math and apply it when they understand it.

We also know children need good visual representations of mathematical concepts. To maintain continuing interest, math must be enjoyable. Math games are a good way to practice basic skills.

Because so many jobs depend on math, we must remedy mathematics education for our own good, our children, our country, and the world.

# Number Sense

This chapter defines number sense as determining and assigning a name to quantities. Topics include counting, subitizing and visualizing, place value, transparent number naming, and writing numbers.

## The Counting Dilemma

Counting is assumed to be the foundation of arithmetic, the door through which every child must pass on the way to basic numeracy. It seems like we've been teaching counting forever. It is so ingrained in our culture that we assume every preschooler is working on their counting and their ABCs. We may have grown up carrying out copious counting activities ourselves. Could counting be less of a foundation than expected? In fact, could counting possibly be a problem?

Parents, grandparents, guardians, daycare providers, teachers and aides, math coaches, textbook writers, program developers, counting-book authors, professors, researchers, and others spend countless hours devising better ways of teaching counting. Many state standards list counting to 100 as the very first standard. This is considered to be part of kindergarten readiness. Children from disadvantaged homes who haven't memorized the list are behind to start and rarely catch up, making this a possible equity issue.

Rarely do people question if counting is the only, or the best way, to learn arithmetic basics. They assume counting is a natural process that would merit the "generally regarded as safe" (GRAS) label. However, these seemingly innocuous practices might not be so harmless after all. Think about how we thought lead in gasoline and asbestos for insulation were great solutions until we learned about their negative side effects.

Some children find counting very difficult. I remember reading the disheartening message written by a mother of a child with Down syndrome who lamented that her child would not be able to learn mathematics because he could not memorize the one-to-ten sequence.

Working with quantities is the foundation of beginning arithmetic; however, one-by-one counting is not the best way to achieve it.

## History of Counting

According to the *American Heritage Dictionary*, the word *count* does not necessarily mean to recite numbers in ascending order. It can also mean to include or determine the total number in a collection. For example, scientists say Pluto no longer counts as a planet. This dual meaning makes it difficult deciphering old texts to determine whether oral counting was actually used.

We do know that for centuries a variety of peoples avoided counting by doing their calculations on abacuses, which were designed to make counting by ones unnecessary. Indigenous peoples in Australia and Brazil have languages that do not have counting words beyond three. Yet, these people are able to perform some basic mathematical tasks.

Some will say that young children love to count; they appear to perform the counting ritual at every opportunity. But are they doing this to please the adult? Or maybe they like the repetition? However, if the counting activity is innate, why don't indigenous children, whose languages do not have counting words, invent words so they can count? Why does it take a child five or six years to become proficient in counting when they master many of the complexities of the English language by age three?

## Counting All and Counting On

One purpose of counting is to teach simple addition. Two sets of objects are given to the child to add, either physical objects or pictures of objects. The child is asked to count the two sets to find the total. Later, the child is given an expression, such as 3 + 2. The child counts out 3 objects and then 2 objects, combines them, and then counts the whole group: 1, 2, 3, 4, 5. This procedure is called *counting all*.

Later, the child is taught to add the expression 9 + 2 by starting at 9 and counting 2 more objects, pictures, or the number words themselves. This process is called *counting on* or *counting up*.

To subtract, children are frequently taught to *count back*. To find 8 – 3, the child is instructed to start at 8, then, count backward three numbers: 7, 6, 5. This type of counting is called *counting back* or *counting down*. Unfortunately, children can become confused and include the 8 when counting backward; 8, 7, 6, producing a wrong answer.

How does counting affect children's later mathematical ability? Educational psychologists study which mathematical practices in the early years contribute to students' later mathematical achievement. According to researchers (Geary et al. 2013), the numerical ability, or number sense, of adolescents was correlated with their knowledge of the number system in kindergarten. But, they found no correlation between adolescents' numerical ability and their kindergarten ability to solve problems by counting. In other words, adolescents' mathematical ability was hampered by their reliance on counting.

## Experiencing Counting as a Child

Parents are so proud when their child can count to 100. This is a memorable achievement. It took considerable time and effort on the part of the child, parents, caregivers, and teachers to memorize that long list of words. Even so, memorizing that long list is only the first of several counting tasks children are expected to master. They must learn that litany so well that they can start and continue from any number. In other words, they can tell you what number comes after 12 or after 39 without starting from one.

To appreciate this difficulty, think of the familiar nursery rhyme, Jack and Jill. Name the word that comes after "hill" *without starting from the beginning*. Challenging! This skill is needed to add by counting on. For example, to add 39 + 3, you must start at 39 and count the next three numbers, 40, 41, 42, without starting at 1.

Next, name the word that comes *before* "hill" in the Jack and Jill rhyme, again without starting from the beginning. Almost

A curious mix of counting the days of school with unrelated computation occurred in a Maryland classroom when a teacher announced, “This is the 82nd day of [school this year], and what would happen if we took away 26?” (Strauss 2005). I wonder why someone would want to take 26 days from the time that has already passed. Do they want to include or exclude weekends? How is this even remotely practical or useful?

## Adding by Counting

The vast majority of children are taught beginning adding by counting. To add 4 + 3, they count 4 objects, then, 3 more objects, and then, count them all. Or, they could start with 4 and count the remaining 3 objects. By doing this numerous times, children are expected to learn their facts.

It is informative to experience this process from a child’s perspective. Let’s do this using the letters of the alphabet instead of the familiar number names. So, A is 1, B is 2, C is 3, and so forth. Now, let’s add F + E.

First, count out F counters. See Figure 1.2.

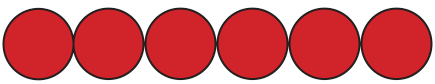


**FIGURE 1.2** F counters

Curiosity question: Did you use one-to-one correspondence to name each dot in Figure 1.2 to check the quantities?

Next, count out E counters. See Figure 1.3.

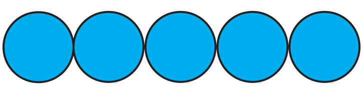


**FIGURE 1.3** E counters

How much is this when you add them all together? Remember that your answer must be a letter, not a number. See Figure 1.4.

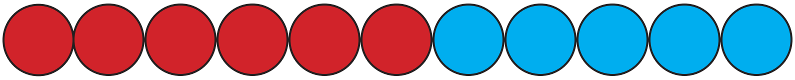


**FIGURE 1.4** F + E counters

Curiosity questions: Did you use the counting all procedure by starting at A and counting to K? Or, did you count on from F: G, H, I, J, K?

How could you add D + C without physical counters? Try using fingers. Start by raising D (four) fingers. Next, extend one finger while saying A; extend another finger while saying B; and one more, saying C. Counting them all gives the answer G. How did you get to G? Did you just know that seven fingers was G, or did you have to count them all?

In addition to counting on with physical objects or fingers, counting on could also be accomplished by counting number words, called *double counting*. For this example, D + C, count on by saying A is E, B is F, and C is G. If you find this burdensome, imagine what it's like for a six-year-old child. Some students have unfortunately resorted to using the 12 numbers on a clock to double count. For example, to add 8 + 6, start with 8, look at the 1 on the clock and say 9, look at the 2 and say 10, look at the 3 and say 11, end at 6 and say 14. The obvious drawbacks to this process include a slow, confusing counting method and the need for an analog clock.

These counting-on activities require a solid knowledge of the number sequence. This explains why kindergarten standards state that students must be able to start at any number in the sequence from 1 to 100 and continue counting, all without starting back at one.

However, not all children worldwide use counting-on procedures. Japanese children are not taught to use counting on; actually, they use few counting procedures for adding or subtracting. More on that later.

Because, you now understand what addition looks like to a child, try memorizing some facts. See the flash cards in Figure 1.5.

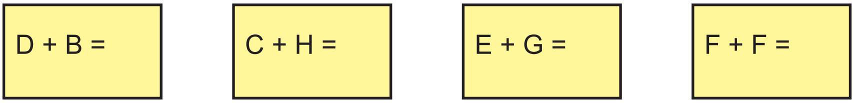



**FIGURE 1.5** Addition flash cards

As you quickly realize, this is a very challenging task. Note that when counting is employed to find sums, place value is simply ignored. It isn't even needed. Fourteen is simply thought of as 14 ones, not as 1 ten and 4 ones, just as N is not thought of as J and D more.

## Subtracting by Counting

Subtracting is considered more difficult than adding because going backward is significantly harder than going forward. Use the alphabet for numbers again. How would you subtract G – D? Start by counting out G dots, as shown in Figure 1.6.

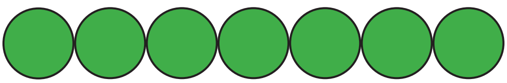


**FIGURE 1.6** G counters

Now, do you move aside one dot while reciting each letter: A, B, C, D, and then, count the remaining dots, A, B, C, to get the difference, or answer, C? See Figure 1.7. Or, did you start at the last counter and count backward four letters: F, E, D, C? A bit challenging, right?

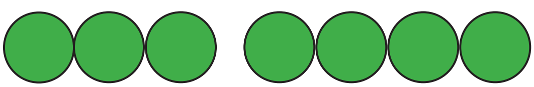


**FIGURE 1.7** G – D counters

Using your fingers to help subtract, you could count backward D times. Raise G fingers and put a finger down each time a letter is recited, F, E, D, C.

Without any counters or fingers, the double-counting procedure is another option. Start at G, and count backward D times. It in long-term memory. I saw this in a school in England where the children were becoming fast counters but were not mastering the facts. Their low scores on a timed test prompted the paraprofessional to chide the pupils for not studying enough. How long would you have to practice to learn the B's facts fluently? Would you still know them a week from now?

## Summing Up

Memorizing the counting words to 100 is an arduous undertaking expected of young children. The added requirement that they must be able to continue from any place in the list makes the chore considerably more burdensome. Many educators expect this assignment to be mastered near the start of kindergarten.

However, many children have disadvantages making this task, more difficult, if not impossible, to meet this goal. Twenty percent of children have memory problems resulting from dyslexia, brain injuries, premature birth, multiple surgeries, or other conditions. Also, they might not have enough support at home. Their parent or guardian might work long hours, have language barriers, be deployed, incarcerated, or simply absent.

Counting itself has downsides. Unfortunately, once counting has been established as the basis for solving basic arithmetic problems, it is a very difficult habit to abandon, although it can be done. Using rote memorization to introduce children to mathematics perpetuates the myth that learning mathematics means lots of memorizing.

Counting does not foster mathematical growth. In fact, it doesn't work in many situations, for example, fractions or money (a coin might have a value of 25). It's even regarded as a bad habit for the older child. Counting is not the center of mathematics any more than the earth is the center of the solar system.

Using counting to instruct young children in mathematics is so ingrained in Western culture that it is difficult to conceive of any other way of teaching mathematics. This is not an excuse to avoid looking for better ways. As an advertisement from Indigo Company states, "We didn't get to the moon by accepting that man can't fly. We didn't get smartphones by stopping at smoke signals. We didn't get the car by accepting the horse. Progress isn't driven by accepting things as they are. It's driven by asking questions" (Indigo 2019). And we need to ask if counting is the true cornerstone of mathematics. We say absolutely not!

## Subitizing and Visualizing

Babies at 5 months of age can add and subtract up to 3. This amazing finding resulted from Karen Wynn's research (1998). In her simplest experiment, Wynn showed a baby two dolls, then, set them on a table in front of the baby. The researcher lowered a screen that hid the dolls. Next, the researcher showed the baby a third doll, and while the baby watched, placed it behind the screen. Then, the screen was raised.

To appreciate what happened next, you need to know that a baby, like us, will look at a novel situation longer than an expected situation. When the baby saw the anticipated three dolls, the baby didn't gaze at them for very long. The experimenter recorded how long the baby looked. For the next trial, the experimenter repeated the process but removed one of the three dolls without the baby's knowledge before the screen was raised. The baby anticipated three dolls, but only saw two, and looked significantly longer as if to say, "I know there were three dolls; where is the missing one?"

Think about what the baby has done. The baby added one doll to the original two dolls. Remember that the baby never saw all three dolls together, and yet they expected to see three. Do you think the baby counted the bears to perform the addition? Highly unlikely. The baby *subitized* the number of dolls and *visualized* them to find the sum.

## Subitizing

The ability to detect quantity without counting is called *subitizing*. This term was coined by Cornelia Coulter at the request of Kaufman and colleagues (1949), who wanted an appropriate new word with no other associations. Their study showed subitizing was a skill distinct from counting or estimating.

# MULTIPLICATION

When thinking of the basic operations of arithmetic, multiplication is often thought of as adding the same number together multiple times. For example, 6 multiplied by 3 is the same as 6 + 6 + 6. Although multiplication is repeated addition at its simplest, the model of repeated addition does not work for multiplying fractions or multiplying negative numbers. Presenting multiplication only as repeated addition will set up students for future challenges and misunderstandings.

Arrays are a better way to introduce multiplication. To find the total number of objects, multiply the number of objects in a row by the number of rows, 6 × 3 = 18. See Figure 5.1.

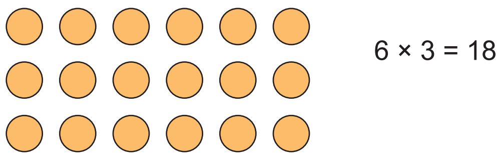



**FIGURE 5.1** Multiplying the number of objects in a row by the number of rows gives the total number, the product.

A common application for multiplication is finding area. When only dimensions are given, as shown in the left rectangle of Figure 5.2, the rote procedure of multiplying the base times the height provides little meaning. However, when square units are drawn in the rectangle, as shown in the right rectangle, it becomes clear that the total number of square units is the number of units in a row (width) times the number of rows (height).


**FIGURE 5.2** Area of the rectangle is 4 × 3 = 12 $cm^2$.

I mentioned to a young lady, who was taking advanced math in high school, that a square was a rectangle. She replied that they had different formulas: the area of a square was $s^2$, while the area of a rectangle was $wh$. Even the perimeters were different: $4s$ and $2w + 2h$. She was astonished to find the formulas were the same when $s$ was replaced with $w$ and $h$. She had never connected the two, as the words were not the same.

Another application of multiplication is used with comparison situations. An example of this is finding the number of hours Morris worked, when Morris worked three times longer than Dana, who worked 2 hours. The expression is $h = 3 \times 2 = 6$ hours.

Also, multiplication solves combination problems, such as how many different outfits are possible with four shirts and two pairs of pants, as shown in Figure 5.3.

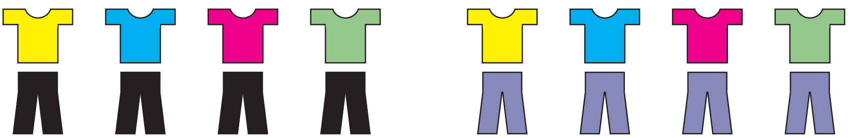


**FIGURE 5.3** Combinations of 4 shirts and 2 pants: 4 × 2 = 8

The usual way to teach beginning multiplication involves the terms *multiplicand*, *multiplier*, and *product*. The multiplicand is the quantity being multiplied; the multiplier is the number of times the quantity is being multiplied, or duplicated; and the product is the answer. Therefore, the expression 6 × 2 means 6 is taken, or repeated, a total of 2 times. Using words, the expression becomes multiplicand × multiplier = product.

Unfortunately, elementary school textbooks often interpret the expression of 6 × 2 as 6 groups of 2. In other words, the 2 is the quantity being operated on, not the 6. This is inconsistent with the other operations of arithmetic. Consider that:

- 6 + 2 means start with 6 and modify it by adding 2.
- 6 – 2 means start with 6 and modify it by removing 2.

- 6 ÷ 2 means start with 6 and modify it by dividing 6 into 2 groups or into groups of 2s.

Consequently, to follow this pattern, 6 × 2 should be interpreted as starting with 6 and modifying it by duplicating 6 a total of 2 times. Part of the confusion results from the meaning of the word "times." The dictionary defines times as "multiplied by." With this definition, 6 × 2 should be read as "6 multiplied by 2," which is consistent with 6 taken 2 times. A visual image of these operations is shown in Figure 5.4.

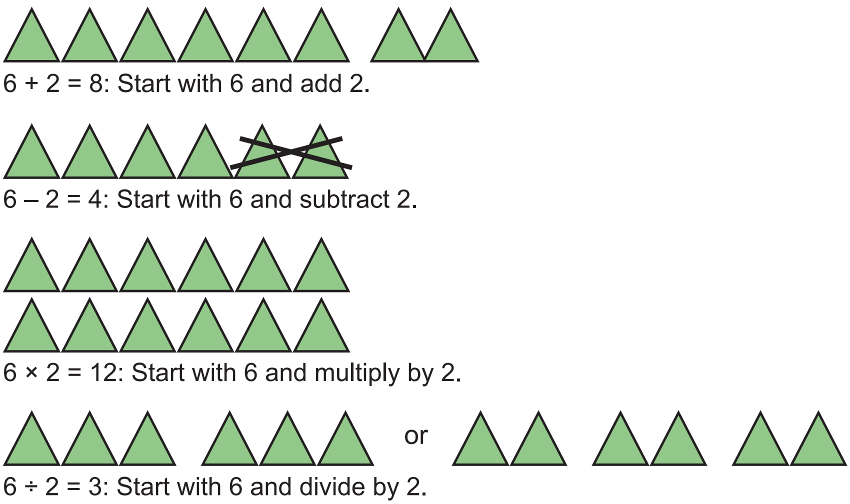



**FIGURE 5.4** The arithmetical operations involving 6 and 2

This confusion over the meaning of the multiplication expression is not a new problem. In 1894, in an attempt to explain the correct meaning, Prince (p. 14) wrote, "In the same way 4 × 2 means that 4 is to be multiplied by 2 and that it be read 4 multiplied by 2."

Note that the sign for multiplication, which has been in use since the 1600s, is not an "x," but "×." It can be thought of as the addition sign, +, turned 45 degrees.

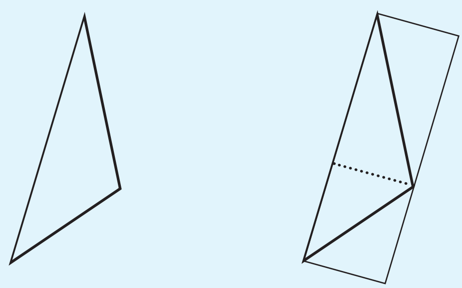
The symbol, ×, is not the only way to indicate multiplication. Programmers and spreadsheet software use the asterisk symbol, *, as in 6 * 2. Sometimes, a middle dot is used, such as 6 • 2. Juxtaposition with no sign means multiplication; for example, *ab* in algebra means *a* and *b* are multiplied together.

My kindergartners enjoyed using the middle dot and the asterisk as substitutions for the traditional multiplication sign.

There seems to be two formulas for finding the area of a rectangle: width × height or base × height. Of course, they are interchangeable and will produce the same results. But, there is a future problem pending when faced with the area of triangles.

Generally, the formula for the area of a triangle is considered to be 1/2 × base × height. There are two concerns with this. First, when presented with the left triangle shown below, where is the base? Second, if the formula for a rectangle is known as width × height, there is little correlation between that formula and the formula of a triangle, other than a height measurement.

If the area of a rectangle is considered to be width × height and the formula for the area of a triangle is 1/2 × width × height, then, it is a quick jump to the realization that the area of a triangle is half the area of a rectangle. This is seen in the second figure. Also, when using the formula of 1/2 × width × height, any of the three sides of the triangle could be the width.



Sometimes, people insist that the multiplicand be represented by concrete objects, and the multiplier is an abstract number of times. This does occur with simple groups, as shown in Figure 5.5 on the next page, where the multiplicand is the three concrete rectangles, and the multiplier is the abstract four times.

Years ago, I tutored Mike, a third grader with learning disabilities. He had no experience with fractions and was moving to a new school in another city. On our last day together at the end of the school year, I suggested we work on fractions because his next class may have already worked on them.

I started by having Mike assemble the fraction puzzle and the stairs. Next, we worked on the fraction names. Then, I asked him to compare 4 and 5, to compare 1/4 and 1/5, and asked how many fourths make a whole. Finally, I explained that 3 one-fourths is written as 3/4.

At the end of the 45-minute session, I said, "Mike, don't think you know everything about fractions. We haven't done something like 1/4 plus 1/8." Mike briefly studied the fraction chart, then announced confidently, "Three-eighths!"

## Fractions Totaling One

Children need to apply the concept that three-thirds or four-fourths is equal to one, or a whole, to fully assimilate it. I was visiting a school where several classes were playing the game, "Concentrating on One" (Cotter 2010, p. 119). In this memory game, players match cards that total one, such as $\frac{2}{5}$ and $\frac{3}{5}$. The children were instructed to look at their fraction chart, find the two-fifths place, and see how many more fifths are needed to make 1, which is three-fifths.

One teacher said her students didn't need their charts. She had taught them a rule that to find the numerator for the matching card, subtract the two numbers on the first card. The result will be the numerator for the needed fraction, while the denominator remains the same. Sadly, these students were not even thinking about fractions; they were playing a simple subtraction game. Of course, it is true that the two numerators will equal the denominator, but that is for the children to discover while they explore and increase their understanding of fractions.

After class, that same teacher told me her children still didn't understand that five-fifths made a whole. I suggested she have her students use the fraction charts. They needed to use the chart to learn and explore the relationships between the fractions.

## Comparing Fractions

Comparing fractions is a major topic in learning fractions. Students are often asked test questions like, "Which is more, $\frac{4}{5}$ or $\frac{5}{6}$?" They're expected to think, using circles, that $\frac{5}{6}$ is a whole circle, less $\frac{1}{6}$, and $\frac{4}{5}$ is a whole circle, less $\frac{1}{5}$. Because $\frac{1}{6}$ is less than $\frac{1}{5}$, that means $\frac{5}{6}$ is greater than $\frac{4}{5}$. How complicated!

A much simpler approach is to refer to the fraction chart. See Figure 7.15.

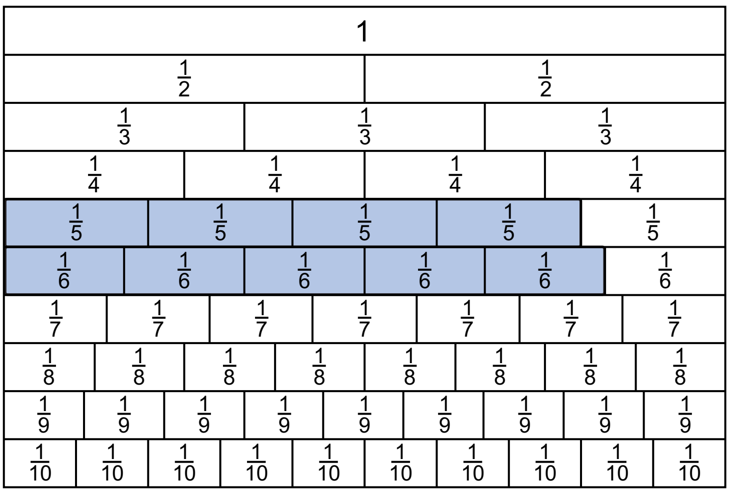
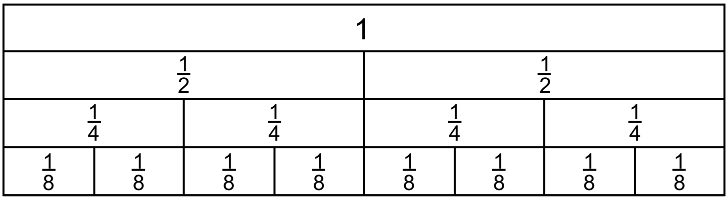


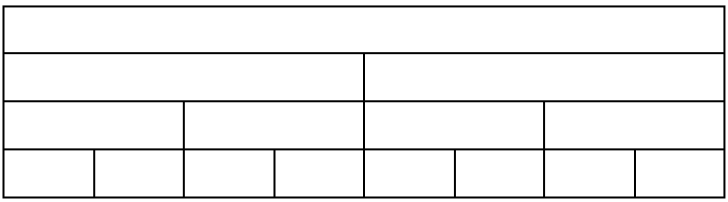

**FIGURE 7.15** Comparing four-fifths and five-sixths

Several strategies are available. These are not rules to be memorized, but observations from the fraction chart. If the denominators are the same, the larger numerator is the larger fraction; for example, $\frac{5}{7}$ is greater than $\frac{2}{7}$. If the numerators are equal, then, the larger denominator signals the smaller fraction; for example, $\frac{3}{10}$ is less than $\frac{3}{8}$. Once the students are thoroughly familiar with the fraction chart, the physical chart will no longer be needed, like the abacus.

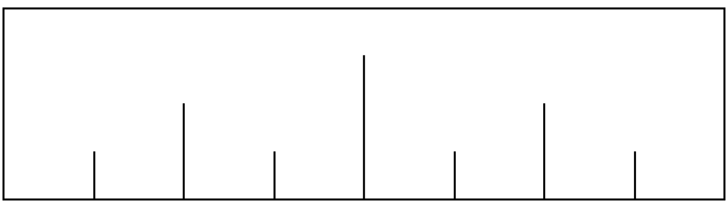
The top figure in Figure 7.16 on the next page shows only the fraction pieces for ones, halves, fourths, and eighths. The next figure shows the same fractions, without the written fractions. The bottom figure shows the horizontal lines removed. Voila, the divisions of an inch on a ruler! Students and adults alike have been amazed to learn how the common ruler is constructed.




The fraction chart with ones, halves, fourths, and eighths.



The written fractions are removed.



The horizontal lines are removed.

**FIGURE 7.16** The source of the inch divisions on a ruler

Children enjoy playing the Fraction War game (Cotter 2010, p. 121), where they compare fractions involving ones, halves, fourths, and eighths, such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$, by referring to the fraction chart, as shown in Figure 7.16. An added bonus for these players is that they learn how to read a ruler.

## Fractions Greater Than One

Much ado is made about fractions being more than one. The historical view of fractions, with its limiting part-of-a-whole mentality, further contributes to the paradigm that a fraction must